Birational cobordisms and factorization of birational maps

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In this paper we develop a Morse-like theory in order to decompose birational maps and morphisms of smooth projective varieties defined over a field of characteristic zero into more elementary steps which are locally étale isomorphic to equivariant flips, blow-ups and blow-downs of toric varieties (see Theorems 1, 2 and 3). The crucial role in the considerations is played by K^*-actions where K is the base field. The importance of K^*-actions in birational geometry and their connection with Mori Theory were already discovered by Thaddeus, Reid and many others (see [Tha1], [Tha2], [Tha3], [R], [D,H]). On the other hand, the ideas of the present paper were inspired by the combinatorial techniques of Morelli's proof of the strong blow-up conjecture for toric varieties ([Mor]).

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The ground field K is assumed to be algebraically closed. All algebraic varieties in this paper and their morphisms are defined over K.

We recall the definitions of good and geometric quotients (see also [Mum]).

Definition 1. Let K^* act on X. By a *good quotient* we mean a variety $Y = X//K^*$ together with a morphism $\pi : X \to Y$ which is constant on G-orbits such that for any affine open subset $U \subset Y$ the inverse image $\pi^{-1}(U)$ is affine and $\pi^* : O_Y(U) \to O_X(\pi^{-1}(U))^G$ is an isomorphism. If additionally for any closed point $y \in Y$ its inverse limit $\pi^{-1}(x)$ is a single orbit we call $Y := X/K^*$ together with $\pi : X \to Y$ a *geometric quotient*.

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Definition 2: Let X_1 and X_2 be two birationally equivalent normal varieties. By a *birational cobordism* or simply a *cobordism* $B := B(X_1, X_2)$ between them we understand a normal variety B with an algebraic action of K^* such that the sets

$$\begin{aligned} B_- : &= \{x \in B \mid \lim_{t\to 0} tx \text{ does not exist}\} \quad \text{and} \\ B_+ : &= \{x \in B \mid \lim_{t\to \infty} tx \text{ does not exist}\} \end{aligned}$$

are nonempty and open and there exist geometric quotients B_-/K^* and B_+/K^* such that $B_+/K^* \simeq X_1$ and $B_-/K^* \simeq X_2$ and the birational equivalence $X_1 - \rightarrow X_2$ is given by the above isomorphisms and the open embeddings $V := B_+ \cap B_-/K^* \subset B_+/K^*$ and $V \subset B_-/K^*$.

Remark. The analogous notion of cobordism of fans of toric varieties was introduced by Morelli in [Mor].

Remark. The above definition can also be considered as an analog of the notion of cobordism in Morse theory. In the present situation, however, a Morse function defining in the classical theory the local action of a 1-parameter group of diffeomorphisms, is replaced by an action of K^*. The objects from Morse theory like bottom and top boundaries, critical points can be interpreted in terms of this action.

Let W be a cobordism in Morse theory of two differentiable manifolds X and X' and $f : W \to [a, b] \subset \mathbf{R}$ be a Morse function such that $f^{-1}(a) = X$ and $f^{-1}(b) = X'$. Then X and X' have open neighbourhoods $X \subseteq V \subseteq W$ and $X' \subseteq V' \subseteq W'$ such that $V \simeq X\times[a, a+\epsilon)$ and $V' \simeq X'\times(b-\epsilon, b]$ for which $f_{|V} : V \simeq X\times[a, a+\epsilon) \to [a, b]$ and $f_{|V'} : V' \simeq X' \times (b - \epsilon, b] \to [a, b]$ are the natural projections on the second coordinate. Let $W' := W \cup_V X \times (-\infty, a+\varepsilon) \cup_{V'} X' \times (b-\varepsilon, +\infty)$. One can easily see that W' is isomorphic to $W \setminus X \setminus X' = \{x \in W \mid a < f(x) < b\}$. Let $f' : W' \to \mathbf{R}$ be the map defined by glueing the function f and the natural projection on the second coordinate. Then $grad(f')$ defines an action on W' of a 1-parameter group $T \simeq \mathbf{R} \simeq \mathbf{R}^*_{>0}$ of diffeomorphisms. The last group isomorphism is given by the exponential.

Set

$$\begin{aligned} W'_- : &= \{x \in W' \mid \lim_{t\to 0} tx \text{ does not exist}\}, \\ W'_+ : &= \{x \in W \mid l\,im_{t\to \infty} tx \text{ does not exist}\}. \end{aligned}$$

Then one can see that W'_- and W'_+ are open and X and X' can be considered as quotients of these sets by T. The critical points of the Morse function are T-fixed points.

Example 1 (Atiyah [A] and Reid [R]). Let K^* act on the $(l+m)$-dimensional affine space $B := A^{l+m}_K$ by

$$t(x_1, ..., x_l, y_1, ..., y_m) = (t \cdot x_1, ..., t \cdot x_l, t^{-1} \cdot y_1, ..., t^{-1} \cdot y_m).$$

Set $\overline{x} := (x_1, ..., x_l)$, $\overline{y} = (y_1, ..., y_m)$. Then

$$\begin{aligned} B_- &= \{(\overline{x}, \overline{y}) \in A^{l+m}_K \mid \overline{y} \neq 0\}, \\ B_+ &= \{(\overline{x}, \overline{y}) \in A^{l+m}_K \mid \overline{x} \neq 0\}. \end{aligned}$$

One can easily see that $B//K^*$ is the affine cone over the Segre embedding of $\mathbf{P}^{l-1} \times \mathbf{P}^{m-1} \to \mathbf{P}^{l\cdot m-1}$, and B_+/K^* and B_-/K^* are smooth.

The relevant birational map $\phi : B_-/K^* \dashrightarrow B_+/K^*$ is a flip for $l, m \geq 2$ replacing $\mathbf{P}^{l-1} \subset B_-/K^*$ with $\mathbf{P}^{m-1} \subset B_+/K^*$. For $l = 1, m \geq 2$, ϕ is a blow-down, and for $l \geq 2, m = 1$ it is a blow-up. If $l = m = 1$ then ϕ is the identity.

Remark. In Morse theory we have an analogous situation. In cobordisms with one critical point we replace S^{l-1} by S^{m-1}.

The following example is a simple generalization of Example 1.

Example 2 (Morelli [Mor]).

Let K^* act on $B := A_K^{l+m+r}$ by

$$t(x_1, ..., x_l, y_1, ..., y_m, z_1, ..., z_r) = (t^{a_1} \cdot x_1, ..., t^{a_l} \cdot x_l, t^{-b_1} \cdot y_1, ..., t^{-b_m} \cdot y_m, z_1, ..., z_r).$$

where $a_1, ..., a_l, b_1, ..., b_m > 0$. Set $\overline{x} = (x_1, ..., x_l), \overline{y} = (y_1, ..., y_m)$, $\overline{z} = (z_1, ..., z_r)$. Then

$$B_- = \{(\overline{x}, \overline{y}, \overline{z}) \in A_K^{l+m+r} \mid \overline{y} \neq 0\},$$
$$B_+ = \{(\overline{x}, \overline{y}, \overline{z}) \in A_K^{l+m+r} \mid \overline{x} \neq 0\}$$

B, B_- and B_+ can be considered as toric varieties acted on by a torus

$$T = \{(\overline{x}, \overline{y}, \overline{z}) \in A_K^{l+m+r} \mid x_i \neq 0, y_j \neq 0, z_k \neq 0 \text{ for any } i, j, k\}.$$

This torus defines the $(l+m+r)$-dimensional lattices $N := \mathrm{Hom}_{alg.gr}(K^*, T)$, $M := \mathrm{Hom}_{alg.gr}(T, K^*)$ and the vector spaces $N_{\mathbf{Q}} = N \otimes \mathbf{Q}$, $M_{\mathbf{Q}} = N \otimes \mathbf{Q}$. For any vectors $v \in N$ and $w \in M$ let t_v denote the corresponding $1-$parameter subgroup and x_w denote the corresponding character. We have a perfect pairing of lattices

$$\langle *, * \rangle : M \times N = \mathrm{Hom}_{alg.gr}(T, K^*) \times \mathrm{Hom}_{alg.gr}(K^*, T) \to \mathrm{Hom}_{alg.gr}(K^*, K^*) \simeq \mathbf{Z}$$

given by $t^{\langle w, v \rangle} = x_w(t_v)$.

Then the action of K^* on the relevant varieties determines a 1-parameter subgroup of T which corresponds to a vector $v_0 \in N$. This defines a projection $\pi : N_{\mathbf{Q}} \to N'_{\mathbf{Q}}$ where $N'_{\mathbf{Q}} := N_{\mathbf{Q}}/(\mathbf{Q} \cdot v_0) \supset N' := N/(\mathbf{Q} \cdot v_0 \cap N)$. The dual vector space to $N'_{\mathbf{Q}}$ is $M'_{\mathbf{Q}} = \{m \in M_{\mathbf{Q}} \mid \langle m, v_0 \rangle = 0\} \supset M' := \{m \in M \mid \langle m, v_0 \rangle = 0\}$.

Consequently, $B \supset T$ is an affine toric variety corresponding to a regular cone $\Delta \subset N_{\mathbf{Q}}$ and B_- (respectively B_+) corresponds to the fan Δ_+ (respectively Δ_-) consisting of the faces of Δ visible from above (respectively below).

Indeed,

$$p \in B_- \equiv \lim_{t \to 0} tp \text{ does not exist}$$
$$\equiv \exists F \in \Delta^\vee \text{such that } \lim_{t \to 0} x_F(tp) = \lim_{t \to 0} x_F(t_{v_0}) x_F(p) \text{ does not exist}$$
$$\equiv \exists F \in \Delta^\vee \text{ such that } \lim_{t \to 0} x_F(t_{v_0}) = \lim_{t \to 0} t^{(F, v_0)} \text{does not exist and}$$
$$x_F(p) \neq 0 \equiv \exists F \in \Delta^\vee \text{ such that } \langle F, v_0 \rangle < 0 \text{ and } p \in X_{\sigma_F}$$
$$\text{where } \sigma_F = \{v \in N_{\mathbf{Q}} \mid \langle F, v \rangle = 0\} \equiv p \in X_{\Delta_+}.$$

Analogously $B_+ = X_{\Delta_-}$(see also [Jur], Thm. 1.5.3).

The quotients B_+/K^* , B_-/K^* and $B//K^*$ are toric varieties corresponding to the fans $\pi(\Delta_+) = \{\pi(\sigma) \mid \sigma \in \Delta^+\}$, $\pi(\Delta_-) = \{\pi(\sigma) \mid \sigma \in \Delta^-\}$ and $\pi(\Delta)$ respectively.

Indeed, $B//K^* = \operatorname{Spec} K[\Delta^\vee \cap M']$ corresponds to the cone

$$(\Delta^\vee \cap M'_{\mathbf{Q}})^\vee = \{w \in M'_{\mathbf{Q}} \mid \langle w, * \rangle_{|(\Delta+\mathbf{Q}\cdot v_0)} \geq 0\}^\vee = (\Delta + \mathbf{Q}\cdot v_0)/\mathbf{Q}\cdot v_0 = \pi(\Delta) \subset N'_{\mathbf{Q}}.$$

Analogously $B_- = X_{\Delta_+}$ (or $B_+ = X_{\Delta_-}$) is obtained by glueing together the affine pieces $X_{\pi(\sigma)}$ where $\sigma \in \Delta_+$ (or respectively $\sigma \in \Delta_-$). Obviously the projection $\Delta^+ \to \pi(\Delta^+)$ is 1-1 and the image $\pi(\Delta^+)$ is a fan.

Note that B_+/K^* and B_-/K^* admit cyclic singularities. This follows from the fact that B_+/K^* is covered by the open sets

$$U_i := \{(\overline{x}, \overline{y}, \overline{z}) \in A_K^{l+m+r} \mid x_i \neq 0\}/K^* \simeq \{(\overline{x}, \overline{y}, \overline{z}) \in A_K^{l+m+r} \mid x_i = 1\}/\Gamma$$

where $\Gamma = \{t \in K^* \mid t(x_i) = t^{a_i}x_i = x_i\}$.

The relevant birational map $\phi : B_-/K^* \dashrightarrow B_+/K^*$ for $l, m \geq 2$ is a toric flip associated with a bistellar operation replacing the triangulation $\pi(\Delta_-)$ of the cone $\pi(\Delta)$ with $\pi(\Delta_+)$. It replaces the product of $A_K^r = \{0, 0, \overline{z}) \in A_K^{l+m+r}\}$ and the $(l-1)$-dimensional weighted projective space defined by the action of T on the $\overline{x}$-coordinates of A_K^{l+m+r} with the product of A_K^r and the $(m-1)$-dimensional weighted projective space defined by the action of T on the $\overline{y}$-coordinates of A_K^{l+m+r}. For $l = 1, m \geq 2$, ϕ is a toric blow-up whose exceptional fibers are weighted projective spaces. For $l \geq 2, m = 1$, ϕ is a toric blow-down. If $l = m = 1$ then ϕ is the identity.

Remark. We prove in Theorems 1 and 2 that if $char K = 0$ then any birational map of any smooth projective or complete varieties can be decomposed into a sequence of toroidal flips, toroidal blow-ups and blow-downs which are locally étale isomorphic to toric flips, blow-ups, and blow-downs described in Example 2.

Definition 3. Let X_1 and X_2 be two birationally equivalent normal varieties and let $\varphi_1 : X_1 \to Y$ and $\varphi_2 : X_2 \to Y$ be two morphisms commuting with the birational equivalence. By a *birational cobordism over* Y between them we understand a cobordism $B := B(X_1, X_2)/Y)$ with a K^*-equivariant morphism $\phi : B \to Y$ where Y is equipped with the trivial K^*-action and such that the following diagrams commute:

$$\begin{array}{cccc} & B_+/K^* & \simeq & X_1 \\ & \uparrow & \searrow & \downarrow \\ \phi_{|B_+}: & B_+ & \to & Y \end{array}$$

$$\begin{array}{cccc} & B_-/K^* & \simeq & X_2 \\ & \uparrow & \searrow & \downarrow \\ \phi_{|B_-}: & B_- & \to & Y \end{array}$$

We say that the cobordism B over Y is *trivial* over an open subset $U \subset Y$ iff there exists an equivariant isomorphism $\phi^{-1}(U) \simeq U \times K^*$, where the action of K^* on $U \times K^*$ is given by $t(x,s) = (x,ts)$.

We shall construct, in terms of the group action, an order on the set of connected components of the fixed point set which corresponds to the order on the set of critical points defined by a Morse function in Morse theory.

Definition 4. Let B be a cobordism. We say that a connected component F of the fixed point set is *an immediate predecessor* of a component F' iff there exists a non-fixed point x such that $\lim_{t\to 0} tx \in F$ and $\lim_{t\to\infty} tx \in F'$. We say that F *precedes* F' and write $F < F'$ if there exists a sequence of connected fixed point set components $F_0 = F, F_1, ..., F_l = F'$ such that F_{i-1} precedes F_i (see [B-B,S], Def. 1.1). We call a cobordism *collapsible* (see also [Mor]) iff the relation $<$ on its set of connected components of the fixed point set is an order. (Here an order is just required to be transitive.)

Remark. The concept of collapsibility in the toric situation was introduced by Morelli in [Mor].

Defintion 5. A cobordism B is *projective* if B is a quasiprojective variety.

Lemma 1. A projective cobordism is collapsible.

Proof. By Sumihiro we can embed the variety equivariantly into a projective space ([Sum], Thm. 1). Each connected component of the fixed point set in the given variety is contained in an irreducible component of the fixed point set of the projective space. The homogeneous coordinates on $\mathbf{P}^n$ can be chosen to be semi-invariants in such a way that

$$t([x_0,...,x_n]) = [t^{a_0}x_0, ..., t^{a_0}x_{l_0}, t^{a_1}x_{l_0+1}, ..., t^{a_k}x_{l_{k-1}+1}, ..., t^{a_k}x_n]$$

, where $0 = a_0 < a_1 < ... < a_k$.

The fixed point components have the following description:

$F_j = \{x \mid x_i = 0 \text{ for all } i \text{ such that } 0 \le i \le l_{j-1} \text{ and } l_j < i \le n\}$

for $j = 0, ..., k$. (We put here $l_1 = -1$.)

We see that for any $a = [a_0, ..., a_n]$,

$$\lim_{t\to 0} ta \in F_j \text{ iff } a_i = 0 \text{ for all } i \le l_{j-1} \text{and } \exists i \le l_j \text{ such that } a_i \ne 0,$$
$$\lim_{t\to\infty} ta \in F_j \text{ iff } a_i = 0 \text{ for all } i > l_j \text{ and } \exists i > l_{j-1} \text{ such that } a_i \ne 0.$$

The order on the fixed point components on $\mathbf{P}^n$ is determined by the relation: $F_i < F_j$ iff $a_i < a_j$.

For any fixed point component F on X let $i(F)$ be the index such that $F \subset F_{i(F)}$. The induced order on fixed point components on X satisfies: if $F < F'$ then $i_F < i_{F'}$. The Lemma is proven.

Let X be a variety with an action of K^*. Let $F \subset X$ be a set of fixed points. Then we define

$$F^+(X) = F^+ = \{x \in X \mid \lim_{t \to 0} tx \in F\},\ F^-(X) = F^- = \{x \in X \mid \lim_{t \to \infty} tx \in F\}.$$

Definition 6. Let B be a collapsible cobordism and F_0 be a minimal component. By an *elementary collapse with respect to* F_0 we mean the cobordism $B^{F_0} := B \setminus F_0^-$. By an *elementary cobordism with respect to* F_0 we mean the cobordism $B_{F_0} := B \setminus \bigcup_{F \neq F_0} F^+$.

Proposition 1. Let F_0 be a minimal component of the fixed point set in a collapsible cobordism B. Then the elementary collapse B^{F_0} with respect to F_0 is again a collapsible cobordism, in particular it satisfies:

a) $B_+^{F_0} = B_+$ is an open subset of B.

b) F_0^- is a closed subset of B and equivalently B^{F_0} is an open subset of B.

c) $B_-^{F_0}$ is an open subset of B^{F_0} and $B_-^{F_0} = B^{F_0} \setminus \bigcup_{F \neq F_0} F^+$.

d) The elementary cobordism B_{F_0} is an open subset of B such that

$$B_{F_0 -} = B_{F_0} \setminus {F_0}^+ = B_-$$
$$B_{F_0 +} = B_{F_0} \setminus {F_0}^- = B_-^{F_0}.$$

e) There exist good and respectively geometric quotients $B_{F_0}//K^*$ and $B_-^{F_0}/K^*$ and moreover the natural embeddings $i_- : B_- \subset B_{F_0}$ and $i_+ : B_-^{F_0} \subset B_{F_0}$ induce proper morphisms $i_{-/K^*} : B_-^{F_0}/K^* \to B_{F_0}//K^*$ and $i_{+/K^*} : B_-/K^* \to B_{F_0}//K^*$.

Before proving the above proposition we shall state a few results on K^*-actions.

Definition 7. Let X be a variety acted on by K^*. By a *sink* (resp. a *source*) of X we mean an irreducible component F of the fixed point set such that F^- (resp. F^+) contains an open subset of X.

Lemma 2. Let X be a variety with a K^*-action for which $\lim_{t \to \infty} tp$ exists for any $p \in X$. Then X contains a sink S and for any fixed point $x \in X$ there exists a sequence $x_0 = x, ..., x_l, y_1, ..., y_l$ of points such that $\lim_{t \to 0} ty_i = x_{i-1}$ and $\lim_{t \to \infty} ty_i = x_i$ for any $i = 1, ..., l$ and $x_l \in S$.

Proof . Let X' be an equivariant completion of X (see [Sum], Thm.3). By ([Sum], Thm.2) we can find a projective normal variety X" with an action of K^* and an equivariant birational morphism $\pi : X" \to X'$. Let S" be a sink in X". Note that $\pi(S")$ is a sink in X' and $\pi(S") \cap X$ is a sink in X' if it is non-empty. Let $x \in X$ be a fixed point and let $x' \in X"$ be a fixed point such that $\pi(x') = x$. By ([Sum], Thm 1) we can embed X" into a projective space and consequently find a sequence $x' = x'_0, ..., x'_l, y'_1, ..., y'_l$ in X" as in the statement of Lemma 2. By applying the morphism π we get a sequence $x = x_0, ..., x_l, y_1, ..., y_l$ in X'. Note that if $x_i \in X$

then $y_{i+1} \in X$ since X' is an open invariant neighbourhood of x_i so it contains all orbits having x_i in their closure . On the other hand by the assumption of Lemma 2 if $y_i \in X$ then $x_{i+1} = \lim_{t\to\infty} ty_i \in X$. Finally the above sequence is contained in X.

As a corollary from the proof of the above lemma we get
Lemma 3. Let X be a variety with a K^*-action and with no sink. Then for any $y \in X$ there exists a sequence $x_1, ..., x_{l-1}, y = y_1, ..., y_l$ of points such that $\lim_{t\to 0} ty_i = x_{i-1}$ for $i = 2, ..., l$, $\lim_{t\to\infty} ty_i = x_i$ for any $i = 1, ..., l-1$ and $\lim_{t\to\infty} ty_l$ does not exist.

Proof of Proposition 1.

a) This follows from the fact that $F_0^- \cap B^+ = \emptyset$.

b) Let $\overline{F_0^-}$ denote the closure of F_0^- in B. Let $\overline{F_0^-} = Z_0 \cup ... \cup Z_k$ be the decomposition into irreducible components. Since B_+ is open and $F_0^- \cap B_+ = \emptyset$ we see that $Z_i \cap B_+ = \emptyset$ for any i. In particular $\lim_{t\to\infty} tp$ exists for any $p \in Z_i$. It follows from Lemma 2 that each Z_i has a sink S_i. Note that $S_i \subset F_0$. If not then by ([Kon], Thm. 9), $S_i^-(Z_i)$ contains an open subset $U_i \subset Z_i$ disjoint from F_0^-. This gives $\overline{F_0^-} \subset Z_0 \cup ... \cup (Z_i \setminus U_i) \cup ... \cup Z_k$, which is a contradiction.

On the other hand, if $Z_i \neq S_i^-(Z_i)$ then for any $x \in Z_i \setminus S_i^-(Z_i)$ we can find a connected component F of the fixed point set of B distinct from F_0 such that $\lim_{t\to\infty} tx \in F$. Now it follows from Lemma 2 applied to Z_i that $F < F_0$, which contradicts the assumption. Finally, $\overline{F_0^-} \subset S_0^-(Z_0) \cup ... \cup S_k^-(Z_k) \subset F_0^-$, which means, that F_0^- is closed.

c) We find directly from the definition of B^{F_0} that $B_-^{F_0} = B^{F_0} \setminus \bigcup_{F\neq F_0} F^+ = B \setminus (\bigcup_{F\neq F_0} F^+ \cup {F_0}^-)$.

By repeating the argument in b) we see that $\overline{F^+}$ consists of points belonging to some components F'^+ of the fixed point set such that $F' \geq F$. Hence $B_-^{F_0} = B^{F_0} \setminus \bigcup_{F\neq F_0} \overline{F^+}$ and is open.

d) The same reasoning as above.

e) Set $U_0 := B_{F_0} \setminus F_0^- \setminus F_0^+ \subset B_-$. Each orbit is closed in U_0. Since B_-/K^* exists we deduce that U_0/K^* exists. Let $U_1, ..., U_s$ be open affine invariant varieties covering F_0. Then $U_0, U_1, ..., U_s$ cover B_{F_0}.

First we prove that

(*) each non-closed orbit is contained in some closed invariant set of the form $\{x\}^+ \cup \{x\}^-$ for some $x \in F_0$.

This is equivalent to

$$\text{for any}\, y \in F_0,\ \{y\}^+ \cup \{y\}^- \cap U_i \neq \emptyset \ \text{ iff } \ (\{y\}^+ \cup \{y\}^-) \subset U_i.$$

It suffices to prove that

$$(U_i \cap F_0)^+ \cup (U_i \cap F_0)^- = U_i \cap (F_0^+ \cup F_0^-).$$

Observe first that for any closed subset $S \subset F_0$ the sets S^- and S^+ are closed in B_{F_0}. Let $\overline{S^-}$ be the closure of S^- and let $\overline{S^-} = Z_0 \cup ... \cup Z_k$ be its minimal decomposition into irreducible components. As in the proof of b) we can show that each Z_i has a sink S_i. Similarly if $Z_i \neq S_i^-(Z_i)$ then it follows from Lemma 2 that $F_0 < F_0$, which is a contradiction. Hence $Z_i = S_i^-(Z_i)$. By the above we get

$$S^- = (S_0 \cap S)^-(Z_0) \cup ... \cup (S_k \cap S)^-(Z_k) \text{ and so}$$
$$\overline{S^-} = \overline{(S_0 \cap S)^-(Z_0)} \cup ... \cup \overline{(S_k \cap S)^-(Z_k)}.$$

The last equality implies $Z_i = \overline{(S_i \cap S)^-(Z_i)}$. This means by ([Kon], Thm. 9) that $S_i \cap S = S_i$ or equivalently $S_i \subseteq S$. Finally, $\overline{S^-} = (S_0)^-(Z_0) \cup ... \cup (S_k)^-(Z_k) \subseteq S^-$, which means that S^- is closed.

It follows that $(U_i \cap F_0)^+ \cup (U_i \cap F_0)^- = F_0^+ \cup F_0^- \setminus (F_0 \setminus U_i)^+ \setminus (F_0 \setminus U_i)^-$ is open in $(F_0^+ \cup F_0^-) \cap U_i$.

Let $p_i : U_i \to U_i//K^*$ for $i = 0, ..., s$ denote the standard projections. Then $p_i(U_i \cap F_0)$ are closed in $U_i//K^*$, and thus $p_i^{-1}p_i(U_i \cap F_0) = (U_i \cap F_0)^+ \cup (U_i \cap F_0)^-$ are closed in U_i which means by the connectedness of F_0 that $(U_i \cap F_0)^+ \cup (U_i \cap F_0)^- = U_i \cap (F_0^+ \cup F_0^-)$. Thus (*) is proven.

Now since the fibers of p_i are single orbits which are closed in B_{F_0} or are of the form $\{x\}^+ \cup \{x\}^-$ for $x \in F_0 \cap U_i$ then for any $i \neq j$ the map $\phi_{ij} : U_i \cap U_j//K^* \to U_i//K^*$ is a bijection and since the above varieties are normal it is an open embedding. So we are in a position to define a good quotient $B_{F_0}//K^*$ as a prevariety by glueing together $U_i//K^*$ along $(U_i \cap U_j)//K^*$. It suffices to prove that $B_{F_0}//K^*$ is separated.

The closed embedding $F_0 \subset B_{F_0}$ defines a map $F_0 \to B_{F_0}//K^*$. This map is bijective since each fiber of the quotient map meeting F_0 is of the form $\{x\}^+ \cup \{x\}^-$ for $x \in F_0$. Moreover for any $i = 0, 1, ..., s$ the map $F_0 \cap U_i \to U_i//K^*$ is a closed embedding. This implies that $F_0 \to B_{F_0}//K^*$ is a closed embedding. Let us identify F_0 with a closed subset in $B_{F_0}//K^*$.

The open embedding $B_- \subset B_{F_0}$ defines a morphism $\phi : B_-/K^* \to B_{F_0}//K^*$ whose restriction to $(B_-/K^*) \setminus \phi^{-1}(F_0)$ is an isomorphism

$$\phi_{|(B_-/K^*)\setminus\phi^{-1}(F_0)} : (B_-/K^*) \setminus \phi^{-1}(F_0) \to (B_{F_0}//K^*) \setminus F_0.$$

Now let R be any valuation ring and $K_0 \supset R$ be its quotient field. Then we have the induced embedding $\operatorname{Spec} K_0 \hookrightarrow \operatorname{Spec} R$. In order to prove the separatedness of $B_{F_0}//K^*$ we have to show that for any map $f : \operatorname{Spec} K_0 \to B_{F_0}//K^*$ there exists at most one extension $f' : \operatorname{Spec} R \to B_{F_0}//K^*$.

If $f(\operatorname{Spec} K_0) \subset F_0$ then $f'(\operatorname{Spec} R) \subset F_0$ and we are done by the separatedness of F_0.

If $f(\operatorname{Spec} K_0) \not\subset F_0$ then $f(\operatorname{Spec} K_0) \subset (B_{F_0}//K^*) \setminus F_0 \subset B_-/K^*$.

This gives the diagram

$$\begin{array}{ccc}
\operatorname{Spec} K_0 & \xrightarrow{f} & B_-/K^* \\
\downarrow & \overset{f_0}{\nearrow} & \downarrow \\
\operatorname{Spec} R & \xrightarrow{f'} & B_F//K^*
\end{array}$$

It suffices to prove that each morphism f' can be lifted to a morphism f_0, in other words that the morphism ϕ is proper. Then we are done by the separatedness of B_-/K^*.

The question is local so we can assume that $f'(\operatorname{Spec} R) \subset U_i//K^* \subset B_{F_0}//K^*$ for some $i = 0, ..., s$. It suffices to consider the case of $i \neq 0$ and U_i affine. Let $\phi : U_i \hookrightarrow A^n_K$ be a closed embedding into an affine space A^n_K with a linear action of K^*. Let F_A denote the fixed point set of A^n_K.

We have the following commutative diagram:

$$\begin{array}{ccccc}
B_{F_0}//K^* & \supset & U_i//K^* & \hookrightarrow & A^n_K//K^* \\
\uparrow i_{-/K^*} & & \uparrow & & \uparrow t_- \\
B_-/K^* & \supset & (U_i \setminus F_0^+)/K^* & \hookrightarrow & (A^n_K \setminus F_A^+)/K^*
\end{array}$$

The morphism t_- is proper and the horizontal arrows are closed embeddings, hence i_{-/K^*} is proper and $B_{F_0}//K^*$ is separated.

Similarly since $B_-^{F_0} = B_{F_0+}$ is fixed point free we see that B_-/K^* is a prevariety. In order to prove the separatedness of $B_-^{F_0}/K^*$ it is sufficient to prove the separatedness of the morphism $B_-^{F_0}/K^* \to B_{F_0}//K^*$. We can reduce the situation to the commutative diagram of separated varieties

$$\begin{array}{ccccc}
B_{F_0}//K^* & \supset & U_i//K^* & \hookrightarrow & A^n_K//K^* \\
\uparrow i_{+/K^*} & & \uparrow & & \uparrow t_+ \\
B_-^{F_0}/K^* & \supset & (U_i \setminus F_0^-)/K^* & \hookrightarrow & (A^n_K \setminus F_A^-)/K^*
\end{array}$$

Properness of the relevant morphism follows from properness of t_+.

As a corollary from Proposition 1 we get

Lemma 4. Let $F_0, ..., F_k$ be connected fixed point set components in a collapsible cobordism B such that $F_i > F_j$ implies $i > j$ for any $i, j = 1, ..., k$. Then B can be represented as a union of elementary cobordisms

$$B = B_{F_0} \cup_{B_{F_0+}} B^{F_0}_{F_1} \cup_{B^{F_0}_{F_1+}} B^{F_0F_1}_{F_2} \cup ... \cup B^{F_0...F_{k-2}}_{F_{k-1}} \cup_{B^{F_0...F_{k-2}}_{F_{k-1}+}} B^{F_0...F_{k-1}}_{F_k}.$$

Construction of a birational cobordism.

Consider a line bundle E over a normal variety X. Let $s_E : X \to E$ be its zero section. Then E defines a line bundle

$$E^\infty := ((E \setminus s_E(X)) \times (\mathbf{P}^1 \setminus \{0\}))/K^*$$

where the action of K^* on

$$(E \setminus s_E(X)) \times (\mathbf{P}^1 \setminus \{0\})$$

is given by $t(x, y) = (tx, t^{-1}y)$ for $x \in E \setminus s_E(X)$ and $y \in \mathbf{P}^1 \setminus \{0\}$. Here the action of K^* on E is standard and the relevant action on $\mathbf{P}^1 \setminus \{0\}$ is induced by the standard embedding $K^* = \mathbf{P}^1 \setminus \{0\} \setminus \{\infty\} \subset \mathbf{P}^1 \setminus \{0\}$).

Definition 8 ([Nag2]). Let X and X' be birationally equivalent varieties with isomorphic open subsets $X \supset U \simeq U' \subset X'$. Let $\Delta : U \to X \times X'$ be the induced morphism. By the *join* $X * X'$ of X and X' we mean the closed subvariety $\overline{\Delta(U)} \subset X \times X'$.

Now let $X \supseteq U \simeq U' \subseteq X'$ be birationally equivalent normal varieties. Let us identify $U \simeq U' \simeq \Delta(U)$. Denote by $\pi : X * X' \to X$ and $\pi' : X * X' \to X'$ the standard projections.

Lemma 5. Let D, D' be effective Cartier divisors on X and X' respectively such that

$$V := X * X' \setminus (\pi^{-1}(supp(D)) \cup \pi'^{-1}(supp(D'))) \subseteq U.$$

Then the open embeddings:

$$\begin{gathered} V \times K^* \subset O_X(-D) \text{ and} \\ V \times K^* \subset O_{X'}(D')^{\infty}. \end{gathered}$$

(obtained by the natural multiplication by the sections corresponding to D and D') define the separated set

$$L(X, D; X', D') := O_X(-D) \cup_{V \times K^*} O_X(D)^{\infty}.$$

Proof. Let R be a valuation ring with quotient field K_0 and valuation ν. We have to prove that for any morphism $\phi_0 : \operatorname{Spec} K_0 \to L(X, D; X', D')$ we can find at most one morphism $\phi : \operatorname{Spec} R \to L(X, D; X', D')$ which makes the following diagram commutative:

$$\begin{tikzcd} \operatorname{Spec} K_0 \arrow[r, "\phi_0"] \arrow[d] & L(X, D; X', D') \\ \operatorname{Spec} R \arrow[ur, "\phi"'] & \end{tikzcd}$$

If $\phi(\operatorname{Spec} K_0) \subset L(X, D; X', D') \setminus (V \times K^*)$ then we are done since by construction the last set is obviously separated.

So we can assume that $\phi_0(\operatorname{Spec} K_0) \subset V \times K^*$. Let $V_0 := V, V_1, ..., V_s$ (respectively $V'_0 := V, V'_1, ..., V'_{s'}$) be an open covering of X (respectively of X') such that for any $i := 1, ..., l$ ($i' := 1, ..., l'$), V_i (respectively $V'_{i'}$) is affine and $D_{|V_i}$ (resp. $D'_{|V'_{i'}}$) is described by $f_i \in K[V_i]$ (resp.$f'_{i'} \in K[V_{i'}]$).

Set $\psi_{ij} : (V_i \cap V_j) \times K \to V_i \times K$, $\psi_{ij}(u, s_j) = (u, (f_j/f_i)s_j)$. Here by s_i we mean the standard coordinate function on $K = \operatorname{Spec} K[s_i]$.

Then $O_X(-D)$ is obtained by glueing $V_i \times K$ along ψ_{ij}.

Analogously we can obtain $O_{X'}(D)^\infty$ by glueing together $V'_{i'} \times (\mathbf{P^1} \setminus \{0\})$ along $\psi'_{i'j'}$ where, $\psi'_{i'j'}(u, s'_{j'}) = (u, (f'_{j'}/f'_{i'})s'_{j'})$.

Here by $s'_{i'}$ we mean the standard coordinate function on $\mathbf{P^1} \setminus \{0\} = \operatorname{Spec} K[1/s'_{i'}]$.

Let t be the coordinate on $V \times K^* = V \times \operatorname{Spec} K[t, t^{-1}]$. Suppose we have two morphisms $\phi, \phi' : \operatorname{Spec} R \to L(X, D; X', D')$. Then we can assume that

$$\phi : \operatorname{Spec} R \to O_X(-D),$$
$$\phi' : \operatorname{Spec} R \to O_{X'}(D')^\infty.$$

Assume that $\phi(\operatorname{Spec} R) \in V_i \times K \subset O_X(-D)$ for some $i \neq 0$. Since the functions $s_i = s_0/f_i = t/f_i$ and f_i are regular on V_i we have

$$\nu(\phi^*(t/f)) = \nu(\phi_0^*(t/f)) \geq 0.$$

Hence $\nu(\phi_0^*(t)) \geq \nu(\phi_0^*(f)) \geq 0$.

On the other hand, we can assume that $\phi'(\operatorname{Spec} R) \subset V'_{i'} \times (\mathbf{P^1} \setminus \{0\}) \subset O_{X'}(D')^\infty$ for some $i' \neq 0$. Then since $1/s'_{i'} = 1/(s'_0 f'_{i'}) = 1/(t f'_{i'})$ and $f'_{i'}$ are regular on $V'_{i'}$ we have

$$\nu(\phi'^*(1/(f'_{i'} \cdot t))) = \nu(\phi_0^*(1/(f'_{i'} \cdot t))) \geq 0.$$

Hence $\nu(\phi_0^*(t)) \leq -\nu(\phi_0^*(f'_{i'})) \leq 0$. Finally $\nu({\phi_0}^*(f_i)) = \nu({\phi_0}^*(f'_{i'})) = \nu({\phi_0}^*(t)) = 0$.

Now let $p : O_X(-D) \to X$ and $p' : O_{X'}(D')^\infty \to X'$ be the standard projections. Then $p\phi : \operatorname{Spec} R \to X$ and $p'\phi' : \operatorname{Spec} R \to X'$ define a morphism $\overline{p\phi} : \operatorname{Spec} R \to X * X'$.

By the previous considerations and the assumptions

$$\overline{p\phi}(\operatorname{Spec} R) \subseteq \{x \in V_i * V'_{i'} \mid f_i(x) \neq 0, f'_{i'}(x) \neq 0\} \subseteq V$$

and consequently $\phi(\operatorname{Spec} R) \subseteq V \times K^*$ and $\phi'(\operatorname{Spec} R) \subseteq V \times K^*$. But this contradicts the separatedness of $V \times K^*$.

Lemma 5 is proven.

For a morphism $\phi : X \to Y$ and a Cartier divisor D on Y we denote by $\phi^*(D)$ its inverse transform. For any birational morphism $\phi : X \to Y$ of smooth varieties and a Weil divisor D on X we denote by $\phi_*(D)$ its strict transform.

Lemma 6.

A. Let $X \supseteq U \simeq U' \subseteq X'$ be normal and projective and D and D' be ample divisors on X and X' respectively such that

$$V := X * X' \setminus (\pi^{-1}(supp(D)) \cup \pi'^{-1}(supp(D'))) \subseteq U.$$

Then $L(X,D;X',D')$ is quasiprojective.
B. Let $X \supseteq U \simeq U' \subseteq X'$ be smooth and projective. Assume that D is ample on X and $supp(D) \supseteq X \setminus U$. Then $L(X,D;X',0)$ is quasiprojective.
C. Let $\phi : X \to X'$ be a birational morphism of smooth projective varieties. Assume that E is an effective divisor on X such that $-E$ is very ample relative to X'. Let D_X and $D_{X'}$ be very ample divisors on X and X' respectively such that $D_X = \phi^*(D_{X'}) - n \cdot E$ for some $n \in \mathbf{N}$ (see [EGA], II, 4.6.13(ii)). Let $D := D_X - \phi_*(D_{X'}) = \phi^*(D_{X'}) - \phi_*(D_{X'}) - n \cdot E$. Then $L(X,D;X',0)$ is quasiprojective.

Proof. Let $p : O_X(-D) \to X$ and $p' : O_{X'}(D')^\infty \to X'$ be the natural projections. Let $S_0 \subset O_X(-D)$ be the zero section divisor and $S_\infty \subset O_{X'}(D')^\infty$ be the infinity section divisor.

For any Weil divisor D on an open subset $U \subset X$ let $\overline{D}$ denote the closure of D in X.

A. Set $D_0 = p^*(D)$ and $D_1 := p'^*(D')$.

Then D_0 and D_1 are Cartier divisors on $L(X,D;X',D')$ since their supports are closed in $L(X,D;X',D')$.

Observe that

$S_\infty + D_0 + (t) = S_0 + D_1$.

Find a natural number n such that nD and nD' are very ample divisors on X and X' respectively. Now one can easily check that

$$D_L := nD_0 + nS_\infty \simeq nD_1 + nS_0$$

is a base point free divisor and for any curve C in $L(X,D;X',D')$ there exists an effective divisor equivalent to D_L which intersects C. This means that D_L defines a quasifinte morphism $\phi : L(X,D;X',D') \to \mathbf{P}^n$, which by the Zariski theorem ([Zar], [Mum2]) can be extended to a finite morphism $\overline{\phi} : \overline{L(X,D;X',D')} \to \mathbf{P}^n$. Then $\overline{\phi}^*(O(1))$ is ample on $\overline{L(X,D;X',D')}$ (see [Har2], Prop. 4.4), which means that $\overline{L(X,D;X',D')}$ is a projective variety and $L(X,D;X',D')$ is quasiprojective.

B. Let D' be ample on X'. Set $D_0 = p^*(D)$ and $D_1 = p'^*(D')$. Then $S_\infty + \overline{D_0} + (t) = S_0$. Again find n such that nD and nD' are very ample divisors on X and X' respectively. Repeat the reasoning of case A for the divisor

$$D_L := nD_0 + n\overline{D_1} + nS_\infty \simeq n\overline{D_1} + nS_0.$$

C. Set $D_0 := p^*(D)$, $D_1 = p^*(D_X)$ and $D_2 := p'^*(D_{X'})$. Then $\overline{D_1} = \overline{D_2} + D_0$ and $D_0 + S_\infty + (t) = S_0$.

Analogously to case A we conclude that

$$D_L := \overline{D_1} + S_\infty = \overline{D_2} + D_0 + S_\infty \simeq \overline{D_2} + S_0$$

is an ample divisor on the quasiprojective variety $L(X,D;X',0)$.

Proposition 2. **A**. There exists a projective cobordism $B(X, X')$ between any two birationally equivalent normal projective varieties X and X'.
B. There exists a smooth cobordism $B(X, X')$ between any birationally equivalent smooth varieties X and X' over a field of characteristic zero.
B'. There exists a smooth projective cobordism $B(X, X')$ between any birationally equivalent smooth projective varieties X and X' over a field of characteristic zero.
C. For any birational morphism $X \to X'$ of smooth projective varieties over a field of characteristic zero which is an isomorphism over $U \subset X'$, there exists a smooth projective cobordism $B(X, X')/X'$ over X' which is trivial over U.

Proof.

In cases A, B', C one can find divisors D and D' satisfying respectively the conditions A, B, C of Lemma 6. In case B we find divisors D and D' satisfying the conditions of Lemma 5.

A. Let $\overline{L(X, D; X', D')}$ be a K^*-equivariant projective completion of $L(X, D; X', D')$ (see [Sum], Thm. 1). Let $\overline{B(X, D; X', D')}$ be its normalization.

B. Let $\overline{L(X, D; X', 0)}$ be a K^*-equivariant completion of $L(X, D; X', 0)$ ([Sum], Thm. 3). Let $\overline{(B(X, D; X', 0)}$ be its canonical K^*-equivariant resolution (see [Hir] and [B-M]).

B'. Let $\overline{L(X, D; X', 0)}$ be a K^*-equivariant projective completion of $L(X, D; X', 0)$. Let $\overline{B(X, D; X', 0)}$ be its canonical K^*-equivariant resolution .

C. Let $\overline{L(X, D; X', 0)}$ be a K^*-equivariant projective completion of $L(X, D; X', 0)$. Let L' denote the graph of the rational map $\overline{L(X, D; X', D')} \to X'$ and $\overline{B(X, D; X', D')}$ be its canonical K^*-equivariant resolution.

Note that in all cases

$$L(X, D; X', D') \subset \overline{B(X, D; X', D')}.$$

(In cases B, B' and C we put $D' = 0$.)

Let $S_0 \subset O_X(-D) \subset L(X, D; X', D')$ be the zero section divisor and $S_\infty \subset O_{X'}(D')^\infty \subset L(X, D; X', D')$ be the infinity section divisor.

Set

$$B(X, X') := \overline{B(X, D; X', D')} \setminus S_0 \setminus S_\infty.$$

Then $B(X, X')_+ = \{x \in B(X, X') \mid \lim_{t\to\infty} tx \,\text{does not exist}\} = \{x \in B(X, X') \subset \overline{B(X, D; X', D')} : \lim_{t\to\infty} tx \in S_0 \cup S_\infty\} = \{x \in B(X, X') \mid \lim_{t\to\infty} tx \in S_\infty\} = B(X, X') \cap S_\infty^- = O_{X'}(D')^\infty \setminus S_\infty$.

Analogously $B_-(X, X') = O_X(-D) \setminus S_0$

In both cases evidently $B_+/K^* \simeq X'$ and $B_-/K^* \simeq X$.

Remark. The above constructed cobordism B between X and X' is of the form $\overline{B} \setminus X' \setminus X$ where $\overline{B}$ is a complete variety with a K^*-action such that X is its source and X' is its sink. One can prove that each cobordism is of that form (see Lemma

7). This makes the analogy between birational cobordism and cobordism in Morse theory stronger.

Another method of constructing cobordisms in case C was communicated to me by Abramovich.

Let $X_2 \to X_1$ be a projective morphism of two smooth varieties. Let $I \subset O$ be a sheaf of ideals such that $X_2 = Bl_I X_1$ is obtained from X_1 by blowing up of I. Let $W = X_1 \times \mathbf{P}^1$ and let $\pi_1 : W \to X_1$, $\pi_2 : W \to \mathbf{P}^1$ be the standard projections. Let z denote the standard coordinate on $\mathbf{P}^1$ and let I_0 be the ideal of the point $z = 0$ on $\mathbf{P}^1$. Then $I' = {\pi_1}^*(I) + {\pi_2}^*(I_0)$ is an ideal supported on $X_1 \times \{0\}$. Set $W' = Bl_{I'}W$. Then the proper transform $\overline{X_1 \times \{0\}}$ of $X_1 \times \{0\}$ is isomorphic to X_2 and it is a sink of W'. For simplicity we identify it with X_2. We prove that W' is smooth at $X_2 \subset W'$. Let $f_1, ..., f_k, z$ generate the ideal I' at $x \in X_1 \times \{0\}$. Then the completion of local ring of any point y of X_2 is up to linear transform of $f_1, ..., f_k$ equal to $\widehat{O_y} = \widehat{O_x}[[f_1, f_2/f_1, ..., f_k/f_1, z/f_1]]$ where z/f_1 generates the ideal of X_2. Since we know that X_2 is smooth we find that $\widehat{O_x}[[f_1, f_2/f_1, ..., f_k/f_1]]$ is regular and finally since z/f_1 is algebraically independent of elements of $\widehat{O_x}[[f_1, f_2/f_1, ..., f_k/f_1]]$ we conclude that that $\widehat{O_y}$ is regular which gives the smoothness of W' at X_2. Now it is sufficient to apply the canonical resolution and we get a smooth variety with sink X_2 and source X_1. By [B-B] we conclude that X_2^+ is a locally trivial K-bundle and finally $W' \setminus X_1 \setminus X_2$ is a smooth projective cobordism from X_2 to X_1.

Lemma 7. **A**. Let B be a normal variety with a K^*-action with no fixed points and such that the geometric quotient B/K^* exists. Then there exists a normal variety $B^0 = B \cup (B/K^*)$ (respectively $B^\infty = B \cup (B/K^*)$) with a K^*-action such that $B^0//K^* \simeq B/K^* \subset B^0$ is a source in B^0 (respectively $B^\infty//K^* \simeq B/K^* \subset B^\infty$ is a sink in B^∞) and the standard projection $B^0 \to B^0//K^*$ (resp. $B^\infty \to B^\infty//K^*$) is given by $x \in B^0 \longrightarrow \lim_{t\to 0} tx$ ($x \in B^\infty \longrightarrow \lim_{t\to\infty} tx$).

B. Let $B(X, X')$ be a cobordism between X and X'. Then there exists a variety $\overline{B(X, X')} = B(X, X') \cup X \cup X' = B(X, X') \cup_{B(X,X')_+} (B(X, X')_+)^\infty \cup_{B(X,X')_-} (B(X, X')_-)^0$ with source X and a sink X'. If X or X' is complete than $\overline{B(X, X')}$ is also complete.

Proof. **A**. Let K^* act on $B \times K$ by $t(x, s) := (tx, t^{-1}s)$ where $x \in B$ and $s \in K$. This action is fixed point free and consequently the quotient $B^0 := (B \times K)/K^*$ is a prevariety. The morphism $(B \times K)/K^* \to B/K^*$ is separated since its restriction to any open affine invariant $U \subset B$ determines a separated morphism $(U \times K)/K^* \to U/K^*$. This implies the separatedness of B^0. The quotient $(B \times K^*)/K^*$ is isomorphic to B. The morphism $i : B \simeq (B \times K^*)/K^* \to (B \times K)/K^* = B^0$ is a 1-1 morphism of normal varieties and hence it is an open embedding. The action of K^* on $B \times K$ defined by $t(x, s) = (tx, s)$ or equivalently $t(x, s) = (x, ts)$ induces an action on B^0 which extends the given action on B.

Moreover

$$(B \times K)/K^* \setminus i((B \times K^*)/K^*) = B^0 \setminus B = (B \times 0)//K^* \simeq B//K^*.$$

Let $\pi : B \times K \to (B \times K)/K^*$ be the standard projection. Let $\overline{x} = \pi(x, s) \in (B \times K)/K^*$. Then $\lim_{t\to 0} t\overline{x} = \pi(\lim_{t\to 0}(x, st)) = \pi(x, 0) \in (B/K^*) \times \{0\}$.

The above reasoning can be repeated for $B^\infty := (B \times (\mathbf{P}^1 \setminus \{0\}))/K^*$.

B. By A we can construct the prevariety $\overline{B(X, X')} = (B_+)^\infty \cup_{B_+} B \cup_{B_-} (B_-)^0$ as in the statement of the Lemma. We prove that this prevariety is separated. We show first that $(B_+)^\infty \cup_{B_+} B$ is separated. It is sufficient to show that $\Delta(B_+) \subset (B_+)^\infty \times B$ is closed. Let $\pi_1 : \overline{\Delta(B_+)} \to (B_+)^\infty$ and $\pi_2 : \overline{\Delta(B_+)} \to B$ denote the standard projections. They are birational morphisms which are isomorphisms over B_+. Suppose on the contrary that there exists $y \in \overline{\Delta(B_+)} \setminus \Delta(B_+)$. In particular $\pi_1(y) \in (B_+)^\infty \setminus B_+)$ belongs to the sink in $(B_+)^\infty$. The points p from the set $\Delta(B_+) \subset \overline{\Delta(B_+)}$ have neither of the two limits $\lim_{t\to 0} tp$ and $\lim_{t\to\infty} tp$ in $\overline{\Delta(B_+)}$. In particular $\overline{\Delta(B_+)}$ has no sink. By Lemma 3 we find a sequence $x_1, ..., x_{l-1}, y = y_1, ..., y_l$ of points in $\overline{\Delta(B_+)}$ such that $\lim_{t\to 0} ty_i = x_{i-1}$ for $i = 2, ..., l$ and $\lim_{t\to\infty} ty_i = x_i$ for $i = 1, ..., l-1$ and $\lim_{t\to\infty} ty_l$ does not exist. By the previous remark $\pi_1(y_l)$ also belongs to a sink of $(B_+)^\infty$. This means that $\lim_{t\to\infty} t\pi_2(y_l)$ does not exist. But this implies that $\pi_2(y_l) \in B_+$ and finally $y_l \in \Delta(B_+)$ and $\pi_1(y_l) \in B_+$, a contradiction. We have proved that $(B_+)^\infty \cup_{B_+} B$ is separated.

Similarly one can prove that $\overline{B(X, X')} = ((B_+)^\infty \cup_{B_+} B) \cup_{B_-} (B_-)^0$ is separated.

Note that $\lim_{t\to 0} tx$ and $\lim_{t\to\infty} tx$ exist for any $x \in \overline{B(X, X')}$. Now assume that X' (or X) is complete. Let $\overline{B(X, X')}'$ be the normalization of a completion of $\overline{B(X, X')}$. Then X is a sink in $\overline{B(X, X')}'$. By Lemma 2 for any $x \in \overline{B(X, X')}'$ we can find a sequence $x_0 = x, ..., x_l, y_1, ..., y_l$ in $\overline{B(X, X')}'$ such that $\lim_{t\to 0} ty_i = x_{i-1}$, $\lim_{t\to\infty} ty_i = x_i$ for any $i = 1, ..., l$ and $x_l \in X$. As in the proof of Lemma 2 we can show that $x \in \overline{B(X, X')}$, which means that $\overline{B(X, X')} = \overline{B(X, X')}'$ is complete.

Lemma 8. Let B_{F_0} be a smooth elementary cobordism. Then for any $x \in F_0$ there exists an invariant neighbourhood V_x of x and a K^*-equivariant étale morphism $\phi : V_x \to T_x$, where $T_x \simeq A^n_k$ is the tangent space with the induced linear K^*-action, such that in the diagram

$$\begin{array}{ccccc}
V_x//K^* \times_{T_x//K^*} T_{x-}/K^* & \simeq & V_{x-}/K^* & \to & T_{x-}/K^* \\
 & & \downarrow & & \downarrow \\
 & & V_x//K^* & \to & T_x//K^* \\
 & & \uparrow & & \uparrow \\
V_x//K^* \times_{T_x//K^*} T_{x+}/K^* & \simeq & V_{x+}/K^* & \to & T_{x+}/K^*
\end{array}$$

the vertical arrows are defined by open embeddings and the horizontal morphisms are defined by ϕ and are étale.

Proof. By taking local semi-invariant parameters at the point $x \in F_0$ one can construct an equivariant morphism $\phi : U_x \to T_x \simeq A^n_K$ from some open affine invariant neighbourhood U_x such that ϕ is étale at x. By Luna's Lemma (see [Lu], Lemme 3 (Lemme Fondamental)) there exists an invariant affine neighbourhood

$V_x \subseteq U_x$ of the point x such that $\phi_{|V_x}$ is étale, the induced map $\phi_{|V_x/K^*} : V_x/K^* \to T_x/K^*$ is étale and $V_x \simeq V_x/K^* \times_{T_x/K^*} T_x$. It follows from the last property that $\phi_{|Gy}$ is an embedding for any $y \in V_{x+}$.

Now since V_{x+} is an open invariant subset of the fixed point free B_+ and B_+/K^* exists it follows that V_{x+}/K^* also exists. Again by the Luna Lemma applied to affine neighbourhoods of any $y \in V_{x+}$ and $\phi(y) \in T_x$ we deduce that $\phi_{/K|V_{x+}/K^*} : V_{x+}/K^* \to T_x/K^*$ is étale at each point $y \in V_{x+}/K^*$ and consequently is étale.

Let $\psi_+ : V_{x+}/K^* \to V_x//K^* \times_{T_x//K^*} T_{x+}/K^*$ be the natural map. Let $\pi_1 : V_x//K^* \times_{T_x//K^*} T_{x+}/K^* \to T_{x+}/K^*$ and $\pi_2 : V_x//K^* \times_{T_x//K^*} T_{x+}/K^* \to V_x//K^*$ denote the natural projections. Since $V_x//K^* \to T_x//K^*$ is étale we infer that π_1 is étale. On the other hand, $\phi_{/K|V_{x+}/K^*} = \pi_1\psi_+$ is étale. This implies that ψ_+ is étale and in particular quasifinite. Since it is a birational quasifinite morphism of normal varieties it is an open embedding by the Zariski Main Theorem ([Mum2]).

On the other hand, $i_{/K^*} : V_{x+}/K^* \to V_x//K^*$ is proper by Proposition 1. But $i_{/K^*} = \pi_2\psi_+$ and since π_+ is separated we conclude that ψ_+ is proper (see [Har1], Cor. 4.8 e)). Finally, a proper morphism which is an open embedding is an isomorphism.

Definition 9 (see also [Dan]). A variety X is called *toroidal* iff for any $p \in X$ there exists an open affine neighbourhood U_x and an étale map $\phi : U_x \to X_{\sigma_x}$ into some affine toric variety X_{σ_x}. X is called *quasismooth toroidal* iff σ_x is a simplicial cone for any $x \in X$.

Definition 10 (see also [Oda], [Mor], [Wlo]). Let $\sigma = \langle v_1, ..., v_r \rangle \subseteq N_{\mathbf{Q}} := N \otimes \mathbf{Q} \simeq \mathbf{Q}^k$ be an r-dimensional simplicial cone spanned by integral vectors $v_1, ..., v_r \in N \simeq \mathbf{Z}^k$. Let $\tau = \langle v_1, ..., v_s \rangle \subset \sigma$ be its face and let $\rho \in \mathrm{Relint}(\tau)$ and let v_ρ be the generator of $N \cap \rho$. By a *star subdivision* σ_ρ of σ at ρ we mean the fan whose set of maximal cones is

$$\{\langle v_\rho, v_1, ..., v_{i-1}, \check{v}_i, v_{i+1}, ..., v_s, ..., v_r \rangle \mid 1 \leq i \leq s\}.$$

We call the corresponding birational toric morphism $X_{\sigma_\rho} \to X_\sigma$ a *toric blow-up* of X_σ.

Definition 11. Let X and Y be two quasismooth toroidal varieties. Then a birational morphism $\psi : X \to Y$ is called *a toroidal blow-up* iff the basic set $L \subset Y$ of ψ is irreducible and there is a simplicial cone $\tau \subset N_{\mathbf{Q}}$ and a ray $\rho \in \mathrm{Relint}(\tau)$ such that for any $y \in L$ there exists an open neighbourhood U_y and a commutative diagram

$$\begin{array}{ccccc} & & U_y & \to & X_{\sigma_y} \\ & & \uparrow \psi & & \uparrow bl_{X_{\sigma_y}} \\ U_y \times_{X_{\sigma_y}} X_{\sigma_{y,\rho}} & \simeq & \psi^{-1}(U_y) & \to & X_{\sigma_{y,\rho}} \end{array}$$

where all the horizontal arrows are étale, τ is a face of $\sigma_y \subset N_{\mathbf{Q}}$ and $bl_{X_{\sigma_y}} : X_{\sigma_{y,\rho}} \to X_{\sigma_y}$ is a toric blow-up.

If $\sigma_y = \tau$ for any $y \in L$ then ϕ_y is called a *simple toroidal blow-up.*

Remark. It follows from the definition that the basic set of a toroidal blow-up is a quasismooth toroidal variety, and the basic set of a simple toroidal blow-up is smooth. The exceptional divisor of a simple toroidal blow-up is a locally free bundle whose fibers are toric varieties associated with the fan $\{\pi(\tau') \mid \tau \text{ is a proper face of } \tau\}$ where $\pi : N_{\mathbf{Q}} \to N_{\mathbf{Q}}/\mathbf{Q}\rho$ denotes the standard projection. In particular if Y is smooth the fibers are just weighted projective spaces.

Defintion 12 (see also [Mor], [Wlo]). Let $\sigma = \langle v_1, ..., v_{k+1} \rangle$ be a k-dimensional cone generated by $k+1$ integral vectors $v_1, ..., v_{k+1}$ with a unique relation $\sum a_i v_i = 0$ where $a_i > 0$ for all $1 \leq i \leq l$ and $a_i < 0$ for $l+1 \leq i \leq k+1$ where l is some number $2 \leq l \leq k$. By a *stellar transform* of σ we mean the transformation replacing the subdivision Σ_1 with the set of maximal simplices $\{\langle v_1, ..., \check{v}_i, ..., v_{k+1} \rangle \mid 1 \leq i \leq l\}$ with another subdivision Σ_2 with the set of maximal simplices $\{\langle v_1, ..., \check{v}_i, ..., v_{k+1} \rangle \mid l+1 \leq i \leq k+1\}$.

We call the diagram

$$\begin{array}{ccc} X_{\Sigma_1} & & X_{\Sigma_2} \\ & \searrow \quad \swarrow & \\ & X_\sigma & \end{array}$$

a *toric flip.*

Definition 13. Let X, Y be quasismooth toroidal varieties and Z be any toroidal variety. Then we call a commutative diagram

$$\begin{array}{ccc} X & & Y \\ & \psi_X \searrow \quad \swarrow \psi_Y & \\ & Z & \end{array}$$

a *simple toroidal flip* if X and Y are quasismooth toroidal varieties and Z is a toroidal variety such that the basic sets $L_X \subset Z$ and $L_Y \subset Z$ of ψ_X and ψ_Y respectively coincide and are irreducible and there exists a toric flip

$$\begin{array}{ccc} X_{\Sigma_1} & & X_{\Sigma_2} \\ & \searrow \quad \swarrow & \\ & X_\sigma & \end{array}$$

such that for any z in the basic set $L_X = L_Y$ there exists an open neighbourhood $U_z \subset Z$ and a commutative diagram of morphisms

$$\begin{array}{rccc} U_z \times_{X_\sigma} X_{\Sigma_1} \simeq & \psi_X^{-1}(U_z) & \to & X_{\Sigma_1} \\ & \downarrow \psi_X & & \downarrow \\ & U_z & \to & X_\sigma \\ & \uparrow \psi_Y & & \uparrow \\ U_z \times_{X_\sigma} X_{\Sigma_2} \simeq & \psi_Y^{-1}(U_z) & \to & X_{\Sigma_2} \end{array}$$

where all the horizontal arrows are étale

Remark. It follows from the definition that the basic set $L_X = L_Y$ of ψ_X and ψ_Y is smooth. The exceptional divisors of ψ_X and ψ_Y are locally free bundles whose fibers are quasismooth toric varieties. In particular if X and Y are smooth the fibers are weighted projective spaces .

Lemma 9. Let B_{F_0} be a smooth elementary cobordism. Then the diagram

$$\begin{array}{ccccc} B_{F_0-}/K^* & & & & B_{F_0+}/K^* \\ & \psi_- \searrow & & \swarrow \psi_+ & \\ & & B_{F_0}//K^* & & \end{array}$$

is either a simple toroidal flip such that the fibers of ψ_- and of ψ_+ are weighted projective spaces or

$\bullet\psi_-$ is an isomorphism and ψ_+ is a simple toroidal blow-up whose fibers are weighted projective spaces or

$\bullet\psi_+$ is an isomorphism and ψ_- is a simple toroidal blow-up whose fibers are weighted projective spaces.

Proof. For any $x \in F_0 \subset B_{F_0}$ the semi-invariant local parameters at x determine a linear action on tangent space at x. These tangent spaces are equivariantly isomorphic for all points of F_0 and determine a unique (up to isomorphism) affine space with a linear action. It is sufficient to apply Lemma 8 and cite Example 2.

Lemma 10. The birational equivalence determined by a simple toroidal flip

$$\begin{array}{ccccc} X & & & & Y \\ & \psi_X \searrow & & \swarrow \psi_Y & \\ & & Z & & \end{array}$$

is the composite of a toroidal blow-up $X \widetilde{\times_Z} Y \to X$ and a toroidal blow-down $Y \leftarrow X \widetilde{\times_Z} Y$, where $X \widetilde{\times_Z} Y$ is the normalization of $X \times_Z Y$

Proof. Take $z \in L_X = L_Y$. The commutative diagram

$$\begin{array}{ccccc} & & X \widetilde{\times_Z} Y & & \\ & & \downarrow & & \\ & & X \times_Z Y & & \\ & \swarrow & & \searrow & \\ X & & & & Y \\ & \searrow & & \swarrow & \\ & & Z & & \end{array}$$

defines locally a diagram

$$\begin{tikzcd}[column sep=small]
 & \psi_X^{-1}(U_z) \widetilde{\times_Z} \psi_Y^{-1}(U_z) \arrow[d] & \\
 & \psi_X^{-1}(U_z) \times_Z \psi_Y^{-1}(U_z) \arrow[dl] \arrow[dr] & \\
\psi_X^{-1}(U_z) \arrow[dr] & & \psi_Y^{-1}(U_z) \arrow[dl] \\
 & U_z &
\end{tikzcd} \qquad (*)$$

On the other hand consider the diagram of toric varieties

$$\begin{tikzcd}[column sep=small]
 & X_{\Sigma_1} \widetilde{\times_{X_\sigma}} X_{\Sigma_2} \arrow[d] & \\
 & X_{\Sigma_1} \times_{X_\sigma} X_{\Sigma_2} \arrow[dl] \arrow[dr] & \\
X_{\Sigma_1} \arrow[dr] & & X_{\Sigma_2} \arrow[dl] \\
 & X_\sigma &
\end{tikzcd}$$

It follows from the universal property of the fiber product that $X_{\Sigma_1} \widetilde{\times_{X_\sigma}} X_{\Sigma_2}$ is a normal toric variety whose fan consists of the cones $\{\tau_1 \cap \tau_2 \mid \tau_1 \in \Sigma_1, \tau_2 \in \Sigma_2\}$. The morphisms $X_{\Sigma_1} \widetilde{\times_{X_\sigma}} X_{\Sigma_2} \to X_{\Sigma_1}$ and $X_{\Sigma_1} \widetilde{\times_{X_\sigma}} X_{\Sigma_2} \to X_{\Sigma_2}$ are toric blow-ups. Moreover the above diagram induces the following one:

$$\begin{tikzcd}[column sep=small]
 & X_{\Sigma_1} \widetilde{\times_{X_\sigma}} X_{\Sigma_2} \times_{X_{\Sigma_1} \times_{X_\sigma} X_{\Sigma_2}} \psi_X^{-1}(U_z) \times_Z \psi_Y^{-1}(U_z) \arrow[d] & \\
 & \psi_X^{-1}(U_z) \times_Z \psi_Y^{-1}(U_z) \arrow[dl] \arrow[dr] & \\
\psi_X^{-1}(U_z) \arrow[dr] & & \psi_Y^{-1}(U_z) \arrow[dl] \\
 & U_z &
\end{tikzcd}$$

Now it is sufficient to show that the above diagram coincides with $(*)$. To this end we note that the morphism

$$\psi_X^{-1}(U_z) \widetilde{\times_Z} \psi_Y^{-1}(U_z) \to X_{\Sigma_1} \widetilde{\times_{X_\sigma}} X_{\Sigma_2} \times_{X_{\Sigma_1} \times_{X_\sigma} X_{\Sigma_2}} \psi_X^{-1}(U_z) \times_Z \psi_Y^{-1}(U_z)$$

is proper birational and étale. Both varieties are normal since the completions of local rings of the second variety are normal (see [Mat], Thm. 34). All this yields that the relevant morphism is an isomorphism. The lemma is proven.

As a corollary we get

Lemma 11. Let B_{F_0} be a smooth elementary cobordism. The birational equivalence $B_{F_0-}/K^* \dashrightarrow B_{F_0+}/K^*$ determined by a flip

$$\begin{tikzcd}[column sep=small] B_{F_0-}/K^* \arrow[dr, "\psi_-"'] & & B_{F_0+}/K^* \arrow[dl, "\psi_+"] \\ & B_{F_0}//K^* & \end{tikzcd}$$

is the composite of a toroidal blow-up $\widetilde{B_{F_0-}/K^* \times_{B_{F_0}//K^*} B_{F_0+}/K^*} \to B_{F_0-}/K^*$ and a toroidal blow-down $B_{F_0+}/K^* \leftarrow \widetilde{B_{F_0-}/K^* \times_{B_{F_0}//K^*} B_{F_0+}/K^*}$ whose fibers are weighted projective spaces, where $\widetilde{B_{F_0-}/K^* \times_{B_{F_0}//K^*} B_{F_0+}/K^*}$ is the normalization of $B_{F_0-}/K^* \times_{B_{F_0}//K^*} B_{F_0+}/K^*$.
Proof. This follows from Lemmas 9 and 10 and the analogous fact for toric flips.

Theorem 1. Let $\pi : X \to X'$ be a birational morphism of smooth projective varieties defined over a field of characteristic zero. Assume that π is an isomorphism over $U \subset X'$ .Then one can find a sequence $X_0 = X, X_1, ..., X_k = X'$ of complete varieties with cyclic singularities together with morphisms $\pi_i : X_i \to X'$, which are isomorphisms over U such that for $i = 0, ..., k-1$ either

•$X_{i+1} \to X_i$ is a simple toroidal blow-up commuting with π_i and π_{i+1} and whose fibers are weighted projective spaces. or

•$X_{i+1} \leftarrow X_i$ is a simple toroidal blow-down commuting with π_i and π_{i+1} and whose fibers are weighted projective spaces or

• there exist a toroidal variety Z_i, a morphism $\pi_i^z : Z_i \to X'$ and a diagram

$$\begin{tikzcd}[column sep=small] X_i \arrow[dr, "\psi_i"'] & & X_{i+1} \arrow[dl, "\psi_{i+1}"] \\ & Z_i & \end{tikzcd}$$

which is a simple toroidal flip commuting with π_i, π_i^z and π_{i+1} respectively such that the fibers of ψ_i and ψ_{i+1} are weighted projective spaces.

Proof. By Proposition 2 we can find a smooth cobordism $B = B(X, X')$ over X' which is trivial over U. Let $F_0, ..., F_k$ be connected fixed point set components such that $F_i > F_j$ implies $i > j$. By Proposition 1 and Lemma 9 we see that $B_-^{F_0 \ldots F_i}/K^* = B_{F_i+}^{F_0 \ldots F_{i-1}}/K^*$ differs from $B_-^{F_0 \ldots F_{i-1}}/K^* = B_{F_i-}^{F_0 \ldots F_{i-1}}/K^*$ for $i = 0, ..., k$ by a simple toroidal flip, a simple toroidal blow-up or a simple toroidal blow-down.

As a corollary we get.

Theorem 2. Let X and X' be smooth projective birationally equivalent varieties defined over a field of characteristic zero with isomorphic open subsets $U \subset X$ and $U' \subset X'$. Then one can find a sequence of complete varieties together with isomorphic open subsets $X_0 = X \supset U_0 = U, X_1 \supset U_1, ..., X_k = X' \supset U_k = U'$ with cyclic singularities such that for $i = 0, ..., k-1$ the birational equivalence $X_i \dashrightarrow X_{i+1}$ defines an isomorphism $U_i \simeq U_{i+1}$ and either

$\bullet X_{i+1} \to X_i$ is a simple toroidal blow-up whose fibers are weighted projective spaces or

$\bullet X_{i+1} \leftarrow X_i$ is a simple toroidal blow-down whose fibers are weighted projective spaces. or

• there exists a toroidal variety Z_i and a diagram

$$\begin{array}{ccccc} X_i & & & & X_{i+1} \\ & \psi_i \searrow & & \swarrow \psi_{i+1} & \\ & & Z_i & & \end{array}$$

which is a simple toroidal flip such that the fibers of ψ_i and ψ_{i+1} are weighted projective spaces.

Proof. Let $\overline{X}$ be a smooth resolution of the join $X * X'$. Apply Theorem 1 to the morphisms $\overline{X} \to X$ and $\overline{X} \to X'$.

As a corollary we get

Theorem 3. Let X and X' be smooth projective birationally equivalent varieties defined over a field of characteristic zero with isomorphic open subsets $U \subset X$ and $U' \subset X'$. Then one can find a sequence of complete toroidal quasismooth varieties together with isomorphic open subsets $X_0 = X \supset U_0 = U, X_1 \supset U_1, ..., X_k = X' \supset U_k = U'$ such that for $i = 0, ..., k-1$ the birational equivalence $X_i \dashrightarrow X_{i+1}$ is either a toroidal blow-up or toroidal blow-down defining the isomorphism $U_i \simeq U_{i+1}$ whose fibers are weighted projective spaces.
Proof. Apply Lemma 11 to Theorem 2.

Remark. By the Moishezon Theorem [Moi], which says that each smooth complete variety over an algebraically closed field of characteristic zero can be rendered projective by a sequence of blow-ups with smooth centers, one can prove theorems similar to Theorems 1, 2, 3 on smooth complete varieties.

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